Homework 3 Solutions

Section 3.1

2. (a) This procedure has well-defined input (namely, a positive integer) upon which it works generally, and each step is definite and effective, but the procedure just keeps doubling forever. So the procedure lacks finiteness and output. We would not call this an algorithm.

(b) This procedure lacks well-defined output, because it will always output “m = 1/0.”

(c) This procedure lacks definiteness because the variable $i$ is undefined.

(d) This procedure has well-defined input upon which it works generally, but the steps involved are indefinite. It doesn’t even make sense to talk about effectiveness or output then.

4. I’ll use pseudocode like in the textbook.

```
procedure maxdiff($a_1, a_2, \ldots, a_n$ : integers)
    maxdiff := 0
    for $i := 1$ to $n$
        for $j := 1$ to $n$
            if $maxdiff < a_i - a_j$ then $maxdiff := a_i - a_j$
    return $maxdiff$
```

16. procedure min($a_1, a_2, \ldots, a_n$ : integers)
    $min := a_1$
    for $i := 1$ to $n$
        if $min > a_i$ then $min := a_i$
    return $min$

24. procedure injective($A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}, f : A \to B$ : a function $f$ between finite sets $A$ and $B$)
    injective := TRUE
    if $m = 1$ then return $injective$
    else for $i := 2$ to $m$
        for $j := 1$ to $i - 1$
            if $f(a_j) = f(a_i)$ then $injective := FALSE$
    return $injective$

Section 3.2

8. (a) $2x^2 + x^3 \log x$ is $O(x^4)$, because if $f_1(x)$ and $f_2(x)$ are both $O(g(x))$, respectively, then $f_1 + f_2$ is $O(g(x))$ too. Note that $x^3 \log x$ could not possibly be $O(x^3)$, because $\log x$ is not $O(1)$ (that is, $\log x$ is not a bounded function). Thus $n = 4$ is actually the smallest integer such that $2x^2 + x^3 \log x$ is $O(x^n)$.\]
(b) Note that \( O(3x^5 + (\log x)^4) = O(\max\{3x^5, (\log x)^4\}) = O(3x^5) = O(x^5) \). Thus, \( 3x^5 + (\log x)^4 \) is \( O(x^5) \). On the other hand, \( 3x^5 + (\log x)^4 \) could not possibly be \( O(x^4) \) because \( x^5 \) is not \( O(x^4) \), a fact we discussed in class.

(c) Note that \( \lim_{x \to \infty} (x^4 + x^2 + 1)/(x^4 + 1) = 1 \), so \( (x^4 + x^2 + 1)/(x^4 + 1) = O(1) \). On the other hand, if \( n < 0 \), then \( x^n \) is \( O(0) \), where 0 is the function that is constantly 0. It is easy to check that \( (x^4 + x^2 + 1)/(x^4 + 1) \) is not \( O(0) \) (again, because \( \lim_{x \to \infty} (x^4 + x^2 + 1)/(x^4 + 1) = 1 \) and \( 1 \neq 0 \)). Thus \( f(x) \) is \( O(1) \).

(d) This is similar to the previous problem: since \( \lim_{x \to \infty} (x^3 + 5 \log x)/(x^4 + 1) = 0 \), then \( f(x) \) is \( O(0) \). But then \( f(x) \) is \( O(x^0) = O(0) \) for all \( n < 1 \). In other words, there is no least integer \( n \), instead \( n \) just keeps getting smaller.

14. (a) No
   (b) Yes
   (c) Yes
   (d) Yes
   (e) Yes
   (f) Yes

16. This is true much more generally: if \( f(x) \) is \( O(g(x)) \) and \( g(x) \) is \( O(h(x)) \), then \( f(x) \) is \( O(h(x)) \). I proved this in class, but I'll repeat the proof here.

   Since \( f(x) \) is \( O(g(x)) \) there exists \( C_1 \) and \( k_1 \) such that
   \[
   |f(x)| \leq C_1 |g(x)| \quad \text{for all } x > k_1,
   \]
   and since \( g(x) \) is \( O(h(x)) \) there exists \( C_2 \) and \( k_2 \) such that
   \[
   |g(x)| \leq C_2 |h(x)| \quad \text{for all } x > k_2.
   \]
   Let \( k = \max\{k_1, k_2\} \) and let \( C = C_1 \cdot C_2 \). Then \( k \) and \( C \) are witnesses to the relation \( f(x) \) is \( O(h(x)) \). Indeed, if \( x > k \), then \( x > k_1 \) and \( x > k_2 \), so both the inequalities \( |f(x)| \leq C_1 |g(x)| \) and \( |g(x)| \leq C_2 |h(x)| \). We can plug in the second inequality to the first one to see that
   \[
   |f(x)| \leq C_1 C_2 |h(x)| = C |h(x)| \quad \text{for all } x > k.
   \]

22. In increasing order, we have

\[
\begin{align*}
& (\log n)^3 \\
& \sqrt{n \log n} \\
& n^{99} + n^{98} \\
& n^{100} \\
& (1.5)^n \\
& 10^n \\
& (n!)^2
\end{align*}
\]
24. Upon further review, I don’t like this problem, so don’t worry about it.

42. No, because of issues with the absolute value involved in the definition of big-O. For example: suppose \( f(x) = x \) and \( g(x) = -x \). Then \( f(x) \) is \( O(g(x)) \). However,

\[
2^{f(x)} = 2^x
\]

whereas

\[
2^{g(x)} = 2^{-x} = \frac{1}{2^x}.
\]

It is straightforward to verify that \( 2^{g(x)} \) is \( O(0) \) and that \( 2^{f(x)} \) is NOT \( O(0) \). We conclude that \( 2^{f(x)} \) could not possibly be \( O(2^{g(x)}) \).