PRACTICE Midterm Exam 1 PRACTICE

Name: Solution Set

You have 50 minutes to complete the exam. It is closed everything: no textbook or notes, no calculators, no phones or laptops, no help from your fellow classmates. Where indicated, you must justify your answer to receive full credit. If you have any questions, please ask me for clarifications.
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1. Determine if the following set identities are true or not. Justify your answer.

(a) (3 points) For all sets $A$ and $B$ in a universe $U$, $\overline{A \cap B} = \overline{A} \cap \overline{B}$

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Since $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ do not have the same membership tables, we can find counterexamples to the proposed identity. For instance, let $U = \{a,b\}$, $A = \{a\}$, and $B = \{b\}$. Then $\overline{A \cap B} = \{b\}$, but $\overline{A} \cap \overline{B} = \emptyset$.

(b) (3 points) For all sets $A$, $B$ and $C$, $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$.

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Since $(A \oplus B) \cap C$ and $(A \cap C) \oplus (B \cap C)$ have the same membership tables, the identity is true.
2. (5 points) Give an example of a bijection \( f : \mathbb{N} \to \mathbb{Z} \).

Let \( F(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \left(\frac{n-1}{2}\right) - 1 & \text{if } n \text{ is odd}. \end{cases} \)
3. (a) (3 points) Show that \( f(x) = x \) is not \( O(\log x) \). (Hint: you might start by using L'Hôpital's rule.)

By L'Hôpital's rule,

\[
\lim_{x \to \infty} \frac{\log x}{x} = 0. \quad (A)
\]

Suppose \( f(x) = x \) is \( O(\log x) \). Then there exist \( C \) and \( k \) such that

\[
|\log x| \leq C |\log x| \quad \text{for all } x > k.
\]

But this would imply

\[
\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{\log x}{x} \geq \frac{1}{C} > 0,
\]

which contradicts \((A)\).

(b) (3 points) Show that \( 3^x \) is \( O(2^{x^2}) \).

We need to find \( C \) and \( k \) such that

\[
|3^x| \leq C |2^{x^2}| \quad \text{for all } x > k.
\]

Let's do some algebra:

\[
3^x \leq 2^{x^2} \quad \Rightarrow \quad \log_3 3^x \leq \log_3 (2^{x^2}) \quad \Rightarrow \quad x \leq x^2 \log_3 2 + \log_3 C.
\]

If \( C = 1 \), this inequality is equivalent to

\[
x \leq x^2 \log_3 2.
\]

\[
\frac{1}{\log_3 2} \leq x, \text{ if } x > \frac{1}{\log_3 2}.
\]

So, if we let \( C = 1 \) and \( k = \frac{1}{\log_3 2} \),

it will be true that

\[
|3^x| \leq |2^{x^2}| \quad \text{for all } x > k.
\]

This shows \( 3^x \) is \( O(2^{x^2}) \).
4. (a) (3 points) Give a big-\(O\) estimate for the time complexity of the following procedure by counting the number of elementary operations. In the following piece of pseudocode, \(a_1, a_2, \ldots, a_n\) are positive integers. Count comparison and multiplications, but ignore comparisons and additions in the for loops.

\[
m := 0 
\text{for } i := 1 \text{ to } n 
\quad \text{for } j := i + 1 \text{ to } n 
\qquad m := \max(a_i, a_j, m) 
\begin{align*}
\sum_{i=1}^{n} 2(n-i) &= 2 \sum_{i=1}^{n} n - 2 \sum_{i=1}^{n} i \\
&= 2n^2 - n(n+1) = n^2 - n \\
\text{total operations. This is } &\Theta(n^2). \quad \text{In fact,}
\end{align*}
\]

For each \(i = 1\) to \(n\), we perform \((n-i) \cdot 2\) comparisons and multiplications. Thus, there are \((n-i) \cdot 2\) operations.

(b) (4 points) Let \(f, g\) and \(h\) be three different procedures which each take an integer as input, and which have running times that are \(\Theta(x), \Theta(x^3)\) and \(\Theta(2^n)\), respectively. Give a big-\(O\) estimate for the time complexity of the following procedure, where \(n\) is a positive integer. You can ignore comparisons and additions in the for loops.

\[
D := g(n) 
\text{for } i := 1 \text{ to } n 
\quad \text{for } j := 1 \text{ to } n 
\qquad \text{for } k := 1 \text{ to } n 
\qquad f(i) + f(j) + h(k) + D \\
\text{is } \Theta(n^3). \\
\text{is } \Theta(n^3) + n^3(0(n) + 0(n) + 0(2^n) + 3) \\
\text{is } \Theta(n^32^n). 
\]
5. (7 points) Use induction to prove that $3^n < n!$ if $n$ is an integer greater than 6.

**Base case:** When $n = 7$,

\[ 3^7 = 9 \cdot 9 \cdot 27 = 81 \cdot 27 = 2187 \]

and $7! = 42 \cdot 20 \cdot 6 = 5040$,

so $3^7 < 7!$.

**Inductive step:** Suppose $3^k < k!$ for some $k \geq 6$.

Since $k \geq 6$, $k+1 > 6 > 3$, so

\[ 3^{k+1} = 3 \cdot 3^k < (k+1) \cdot 3^k. \]

By our inductive hypothesis,

\[ (k+1) \cdot 3^k < (k+1) \cdot k! = (k+1)!. \]

Putting this all together,

\[ 3^{k+1} < (k+1)!, \]

which completes our proof by induction.